jednoliko = uniformno

11. 05. 2010.

FUNKCIONALNI NIZOVI

$n \in \mathbb{N}$

$f_1(x), f_2(x), \ldots, f_n(x), \ldots$ ⇒ opći član

$x \in S \subseteq \mathbb{R}$

$\{f_n(x)\}_{n\in\mathbb{N}} \quad x \in S \subseteq \mathbb{R}$

$x = x_0 \Rightarrow \{f_n(x_0)\}_{n\in\mathbb{N}}$ je konstantni niz

Ako niz $\{f_n(x)\}$ konvergira za svako $x \in S$ tada postoji funkcija $y = f(x)$ definisana na skupu S tako da je $\lim\limits_{n\to\infty} f_n(x) = f(x)$

Def. $\lim\limits_{n\to\infty} f_n(x) = f(x)$ ako $(\forall \varepsilon > 0)(\forall x \in S)(\exists n_0 = n_0(\varepsilon, x) \in \mathbb{N})$ $n \geq n_0 \Rightarrow |f_n(x) - f(x)| < \varepsilon$.

piše se: $f_n \xrightarrow[x\in S]{} f$

Def ravnomjerne konvergencije: Niz $\{f_n(x)\}_{n\in\mathbb{N}}$ konvergira ravnomjerno (uniformno) na skupu S ka funkciji $f(x)$ ako: $(\forall \varepsilon > 0)(\forall x \in S)(\exists n_0 = n_0(\varepsilon) \in \mathbb{N})$ $n \geq n_0 \Rightarrow |f_n(x) - f(x)| < \varepsilon$

piše se: $f_n \underset{x\in S}{\rightrightarrows} f$

Dovoljan uslov da $f_n \underset{S}{\rightrightarrows} f$: $(a_n \geq 0 \quad n \in \mathbb{N})$

Ako postoji niz $\{a_n\}_{n\in\mathbb{N}}$, $a_n \to 0$, tako da je $|f_n(x) - f(x)| \leq a_n \quad (x \in S)$ onda $f_n \underset{S}{\rightrightarrows} f$

Potreban i dovoljan uslov da $f_n \underset{S}{\rightrightarrows} f$:

$$\lim_{n\to\infty} \sup_{x\in S} |f_n(x) - f(x)| = 0$$

$\left(\text{ako } f_n \underset{S}{\rightrightarrows} f\right) \Downarrow$

Ako želimo dokazati da $f_n \not\underset{S}{\rightrightarrows} f$, možemo na dva načina:

1° $\lim\limits_{n\to\infty} \sup\limits_{x\in S} |f_n(x) - f(x)| \neq 0$ ili

$(\forall n \geq n_0)$

2° $(\exists\, \varepsilon_0 > 0)(\exists\, n_0 \in \mathbb{N}) \vdots (\exists \{x_n\}_{n\in\mathbb{N}} \subseteq S)$

$|f_n(x_n) - f(x_n)| \geq \varepsilon_0$

PRIMJERI:

1. Ispitati da li dati funkcionalni niz konvergira na datom skupu i ako konvergira naći limes tog niza.

a) $f_n(x) = x^n, \quad S = [0,1]$

$$\lim_{n\to\infty} x^n = \begin{cases} 1, & x = 1 \\ 0, & |x| < 1 \\ +\infty, & x > 1 \end{cases}$$

$$f(x) = \begin{cases} 0, & x \in [0,1] \\ 1, & x = 1 \end{cases}$$

b) $f_n(x) = x^4 \cdot \cos\frac{1}{nx}, \quad S = (0, +\infty)$

$|f_n(x)| = |x^4 \cos\frac{1}{nx}| \leq x^4$

$\lim\limits_{n\to\infty} \cos\frac{1}{nx} = \cos 0 = 1 \qquad f(x) = x^4$

c) $f_n(x) = \ln\left(3 + \frac{n^2 e^x}{n^4 + e^{2x}}\right), \quad S = [0, +\infty)$

$\ln(1+t) \sim t \quad (t \to 0)$ asimptotske jednakosti Analiza I

$f_n(x) = \ln 3\left(1 + \frac{n^2 e^x}{3(n^4 + e^{2x})}\right) = \ln 3 + \ln\left(1 + \frac{n^2 e^x}{3(n^4+e^{2x})}\right) \left[\sim \frac{n^2 e^x}{3n^4 + 3e^{2x}} \sim \right.$

$\left. \sim \frac{n^2 e^x}{3n^4} = \frac{e^x}{3n^2} \to 0 \right] \sim \ln 3 + \frac{e^x}{3n^2} \qquad f(x) = \ln 3 \quad (x \geq 0)$

d) $f_n(x) = n x^n$, $S_1 = [0,1]$ $\quad S_2 = [0, \frac{1}{2}]$

$f_n(x)$ divergira na skupu S_1 jer $f_n(0) = 0$, $f_n(1) = n \xrightarrow{(n\to\infty)} +\infty$

Za skup S_2 :

$f_n(0) = 0$

Neka je $x_0 \in (0, \frac{1}{2}]$. Posmatrajmo pozitivni red: $\sum_{n=1}^{\infty} n x_0^n$

$\sqrt[n]{n \cdot x_0^n} = \sqrt[n]{n} \cdot \sqrt[n]{x_0^n} = \underset{\downarrow 1}{\sqrt[n]{n}} \cdot x_0 \longrightarrow x_0 < 1$

Prema Košijevom korjenovom kriteriju imamo da konvergira red $\sum_{n=1}^{\infty} n x_0^n$, pa zato njegov opći član teži ka 0,

$\lim_{n\to\infty} n x_0^n = 0$, zato je $f(x) = 0$ za $x \in S_2$

(2.) Ispitati ravnomjernu konvergenciju niza na datom skupu.

a) $f_n(x) = x^n - x^{n+1}$, $S = [0,1]$

$f_n(x) = x^n(1-x)$

1. a) $\Rightarrow f(x) = 0 \quad (x \in S)$

$|f_n(x) - \underset{0}{f(x)}| = |f_n(x)| = |\underbrace{x^n}_{\geq 0} \underbrace{(1-x)}_{\geq 0}| = x^n(1-x) = x^n - x^{n+1}$

$\lim_{n\to\infty} \sup_{x\in S} |f_n(x) - f(x)| = \lim_{n\to\infty} \sup_{x\in S} (x^n - x^{n+1}) = ?$

$f_n'(x) = n x^{n-1} - (n+1) x^n = x^{n-1}[n - (n+1)x]$

$f'(x) = 0 \Rightarrow x_1 = 0 \quad , \quad x_2 = \frac{n}{n+1}$

	0	$\frac{n}{n+1}$	
$f_n'(x)$	$+$	0	$-$
$f_n(x)$	$\nearrow$	max	$\searrow$

$n-(n+1)x>0$ $\quad x<\frac{n}{n+1}$

$n>(n+1)x$

$$f_n\left(\frac{n}{n+1}\right)=\left(\frac{n}{n+1}\right)^n\cdot\left(1-\frac{n}{n+1}\right)=\left(\frac{n}{n+1}\right)^n\cdot\frac{1}{n+1}$$

$$\sup_{x\in S}|f_n(x)-f(x)|=\left(\frac{n}{n+1}\right)^n\cdot\frac{1}{n+1}$$

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\frac{1}{n+1}=\lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^n\frac{1}{n+1}=\lim_{n\to\infty}\frac{1}{(1+\frac{1}{n})^n}\cdot\frac{1}{n+1}=\frac{1}{e}\cdot 0=0$$

$$f_n \underset{S}{\rightrightarrows} f$$

6)* $f_n(x)=\operatorname{arctg}\frac{2x}{x^2+n^3}$, $\quad S=\mathbb{R}$

$$\lim_{n\to\infty}\operatorname{arctg}\frac{2x}{x^2+n^3}=\operatorname{arctg}\lim_{n\to\infty}\frac{2x}{n^3}=\operatorname{arctg}0=0$$

$f_n(x)=0$

$$|f_n(x)-f(x)|=|f_n(x)|=\left|\operatorname{arctg}\frac{2x}{x^2+n^3}\right|$$

$$f_n'(x)=\frac{1}{1+\frac{4x^2}{(x^2+n^3)^2}}\cdot\frac{2(x^2+n^3)-2x\cdot 2x}{(x^2+n^3)^2}=\frac{\cancel{(x^2+n^3)^2}}{(x^2+n^3)^2+4x^2}\cdot\frac{2(n^3-x^2)}{\cancel{(x^2+n^3)^2}}=$$

$$=\frac{2(n^3-x^2)}{(x^2+n^3)^2+4x^2}$$

	$-\infty$	$-n\sqrt{n}$	$n\sqrt{n}$	$+\infty$
$f_n'(x)$	$-$	0	$+$	0	$-$
$f_n(x)$	$\searrow$	min	$\nearrow$	max	$\searrow$

funkcija je neparna (modul min i max je jednak)

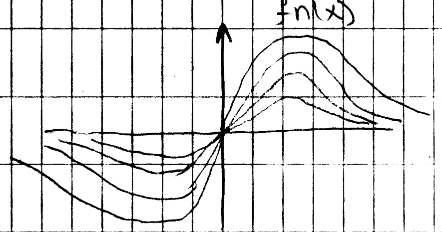
$2(n^3-x^2)=0$

$n^3=x^2$

$x=\pm n\sqrt{n}$

$$f_n(n\sqrt{n}) = \operatorname{arctg} \frac{2n\sqrt{n}}{n^3+n^3} = \operatorname{arctg} \frac{n\sqrt{n}}{n^3} = \operatorname{arctg} \frac{\sqrt{n}}{n^2} = \operatorname{arctg} \frac{1}{n\sqrt{n}}$$

$$\Rightarrow |f_n(x) - f(x)| \leq \operatorname{arctg} \frac{1}{n\sqrt{n}}$$



$$\lim_{n\to\infty} \operatorname{arctg} \frac{1}{n\sqrt{n}} = \operatorname{arctg} \lim_{n\to\infty} \frac{1}{n\sqrt{n}}^{\nearrow 0} = 0$$

$$\lim_{n\to\infty} \sup_{x\in S} |f_n(x) - f(x)| = 0 \qquad f_n(x) \underset{n\to\infty}{\rightrightarrows} f$$

c) $f_n(x) = \dfrac{n^2}{n^2+x^2}$, $S = [-1, 1]$

$$f(x) = \lim_{n\to\infty} \frac{n^2}{n^2+x^2} = \lim_{n\to\infty} \frac{n^2}{n^2} = 1$$

$$|f_n(x) - f(x)| = \left| \frac{n^2}{n^2+x^2} - 1 \right| = \left| \frac{n^2-(n^2+x^2)}{n^2+x^2} \right| = \left| \frac{-x^2}{n^2+x^2} \right|$$

$$= \frac{x^2}{n^2+x^2} \leq \frac{x^2}{n^2} \leq \frac{1}{n^2}$$

$$|x| < 1 \Rightarrow x^2 < 1$$

$$|f_n(x) - f(x)| \leq \frac{1}{n^2} \quad (n\in\mathbb{N}, x\in S) \quad \frac{1}{n^2} \to 0 \quad (n\to\infty)$$

Prema dovoljnom uslovu za ravnomjernu konvergenciju dati red konvergira na skupu S.

d) $f_n(x) = \dfrac{nx}{1+n^2x^2}$, $S = [0, 2]$

$$f(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2} = \lim_{n\to\infty} \frac{nx}{n^2x^2} = \lim_{n\to\infty} \frac{1}{nx} = 0$$

$x_n = \frac{1}{n} \quad (n \in \mathbb{N})$, $x_n \in S \quad (n \in \mathbb{N})$

$$|f_n(x_n) - f(x)| = f_n(x_n) = f_n\left(\frac{1}{n}\right) = \frac{\cancel{n} \cdot \frac{1}{\cancel{n}}}{1 + \cancel{n^2} \cdot \frac{1}{\cancel{n^2}}} = \frac{1}{2}$$

$\varepsilon_0 = \frac{1}{2}$, $f_n \not\overset{S}{\rightrightarrows} f$

e) $f_n(x) = \sin \frac{x}{n}$, $S = \mathbb{R}$

$$f(x) = \lim_{n\to\infty} \sin \frac{x}{n} = \sin \lim_{n\to\infty} \frac{x}{n} = \sin 0 = 0$$

$$|f_n(x) - f(x)| = |f_n(x)| = \left|\sin \frac{x}{n}\right|$$

$\sup \left|\sin \frac{x}{n}\right| = 1$

$\lim \sup \left|\sin \frac{x}{n}\right| = 1$ $\quad f_n \not\overset{S}{\rightrightarrows} f$

Za vježbu:

f) $f_n(x) = \frac{\sin nx}{n}$, $S = \mathbb{R}$

g)* $f_n(x) = \frac{\operatorname{arctg} nx}{\sqrt{n+x}}$ $\quad S = [0, +\infty]$

h)* $f_n(x) = \frac{x + xn^3 + x^3n^6}{1 + x^2n^6}$, $S = [1, +\infty]$

i) $f_n(x) = x^n + x^{2n} - 2x^{3n}$, $S = [0, 1]$

j)* $f_n(x) = nx(1-x)^n$, $\quad S = [0, 1]$

k)* $f_n(x) = \sqrt{x + \frac{1}{n}}$ $S = [0, +\infty)$

Dopuna za 1. zadatak:

e) $f_n(x) = n \sin \frac{1}{nx}$, $S = [0, +\infty)$

f) $f_n(x) = n \cdot (\sqrt[n]{x} - 1)$ $S = [1, 3]$

③ Ispitati ravnomjernu konvergenciju datog niza na datim skupovima.

a) $f_n(x) = \frac{x}{x+n}$, $S_1 = [0, a]$, $a > 0$, $S_2 = [0, +\infty)$

$\lim \frac{x}{x+n} = 0$ → konstanta $f(x) = 0$

$|f_n(x) - f(x)| = |f_n(x)| = f_n(x) = \frac{x}{x+n} \leq \frac{x}{n}$

$x \in S_1, \Rightarrow |f_n(x) - f(x)| \leq \frac{a}{n} \to 0 \quad (n \to \infty)$

$f_n \underset{S_1}{\rightrightarrows} f$

$x_n = n \in S_2 \quad (n = 1, 2, \ldots)$

$|f_n(x_n) - f(x_n)| = |f_n(x_n)| = f_n(x_n) = \frac{n}{n+n} = \frac{n}{2n} = \frac{1}{2}$

$f_n \underset{S_2}{\not\rightrightarrows} f$ (konvergira ali ne ravnomjerno)

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b) $f_n(x) = \frac{x^n}{1 + x^n}$ $S_1 = (0, 1-\delta)$ $S_2 = (1+\delta, +\infty)$ $(0 < \delta < 1)$

(neće ravnomjerno konvergirati) ← $S_3 = (0, 1)$ $S_4 = (1, +\infty)$

$S_5 = (1-\varepsilon, 1+\varepsilon)$ $0 < \varepsilon < 1$

S_1 slično kao S_3

S_2 slično kao S_4

broj 1/2 intervala $[0,1]$ kad se stepenuje sa n je uvjet 0.1

$x \in S_1 \Rightarrow |x| < 1 \Rightarrow \lim\limits_{n\to\infty} x^n = 0$

$f(x) = 0 \quad x \in S_1$

$\underbrace{|f_n(x)}_{\geq 0} - \underbrace{f(x)}_{0}| = f_n(x) = \dfrac{x^n}{1+x^n} \leq x^n \leq (1-\delta)^n$

$(1-\delta) \in [0,1] \Rightarrow \lim\limits_{n\to\infty} (1-\delta)^n = 0 \qquad f_n \underset{S_1}{\rightrightarrows} f$

$x \in S_2 \Rightarrow x > 1 \Rightarrow \lim\limits_{n\to\infty} x^n = +\infty$

$\lim\limits_{n\to\infty} \dfrac{x^n}{1+x^n} \dfrac{1 : x^n}{1 : x^n} = \lim\limits_{n\to\infty} \dfrac{1}{\frac{1}{x^n}+1} = 1 \qquad f(x) = 1 \quad , x \in S_2$

$|f_n(x) - f(x)| = \left|\dfrac{x^n}{1+x^n} - 1\right| = \left|\dfrac{\cancel{x^n} - (1+\cancel{x^n})}{1+x^n}\right| = \dfrac{1}{1+x^n} \leq \dfrac{1}{x^n} < \dfrac{1}{(1+\delta)^n}$

$x > 1+\delta \Rightarrow x^n > (1+\delta)^n \Rightarrow \dfrac{1}{x^n} < \dfrac{1}{(1+\delta)^n}$

$1+\delta > 1 \Rightarrow \lim\limits_{n\to\infty} (1+\delta)^n = +\infty$

$f_n \underset{S_2}{\rightrightarrows} f$

Za skup S_3 , $f(x) = 0$

S_4 : $f(x) = 1$

S_5 : $f(x) = \begin{cases} 0, & x \in (1-\varepsilon, 1) \\ \frac{1}{2}, & x = 1 \\ 1, & x > 1 \end{cases}$

Za ova tri skupa S_3, S_4, S_5 može se dokazati da:

$\lim\limits_{n\to\infty} \sup\limits_{x \in S_i} |f_n(x) - f(x)| = \frac{1}{2} \quad . \quad f_n \underset{S_3,S_4,S_5}{\not\rightrightarrows} f$

Za vježbu :

$\overset{*}{c})\ f_n(x) = \dfrac{nx^2}{1+2n+x} \qquad S_1 = [0,1] \qquad S_2 = [1,+\infty)$

d) $f_n(x) = \frac{x}{n} \ln \frac{x}{n}$ $S_1 = (0, 2)$ $S_2 = (0, +\infty)$

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FUNKCIONALNI REDOVI

$\sum_{n=1}^{\infty} f_n(x) = f_1(x) + f_2(x) + \dots + f_n(x) + \dots$

$x = x_0 \Rightarrow \sum_{n=1}^{\infty} f_n(x_0)$ je brojni red

$S_n(x) = \sum_{k=1}^{n} f_k(x) = f_1(x) + \dots + f_n(x)$

Def $\sum_{n=1}^{\infty} f_n(x)$ konvergira na skupu $S \subseteq \mathbb{R}$ ako konvergira funkcionalni niz $\{S_n(x)\}$ na tom skupu. Kažemo da red $\sum_{n=1}^{\infty} f_n(x)$ konvergira ravnomjerno na skupu $S \subseteq \mathbb{R}$ ako niz parcijalnih suma $\{S_n(x)\}$ tog reda konvergira ravnomjerno na skupu S.
Oblast konvergencije reda $\sum_{n=1}^{\infty} f_n(x)$ je skup $S \subseteq \mathbb{R}$ (to je uglavnom neki interval) takav da za svako $x_0 \in S$ konvergira red $\sum_{n=1}^{\infty} f_n(x_0)$
Dovoljno je ustanoviti intervale na kojima dati funkcionalni red apsolutno konvergira.

Weierstrasse-ov (Vajerštrasov) kriterij

Red $\sum_{n=1}^{\infty} f_n(x)$ konvergira ravnomjerno na skupu $S \subseteq \mathbb{R}$ ako postoji pozitivni brojni red $\sum_{n=1}^{\infty} a_n$ koji konvergira i :

$|f_n(x)| \leq a_n$ za sve $x \in S$ i za sve n dovoljno velike $(n \geq n_0)$

$$\boxed{|t| < r \quad (r>0) \Rightarrow -r < t < r}$$

(1) Odrediti oblast konvergencije datog reda:

a) $\sum_{n=1}^{\infty} e^{-nx}$

$\sqrt[n]{e^{-nx}} = e^{-x}$

Cauchy - korjeni kriterij:

$q = \lim_{n\to\infty} \sqrt[n]{a_n} < 1 \Rightarrow$ red konvergira

$q = e^{-x} < 1 \qquad -x < 0 \Rightarrow \boxed{x>0} \qquad S = (0, +\infty)$

$e^{-x} < e^{0}$

$x = 0 \Rightarrow \sum_{n=1}^{\infty} e^{0} = 1+1+1+\dots$ - divergira (opšti član ne $\to 0$)

b) $\sum_{n=2}^{\infty} \frac{\ln^n x}{n}$

D'Alambert-ov kriterij:

$q = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1 \Rightarrow$ red konvergira

$a_n = \left| \frac{\ln^n x}{n} \right| = \frac{\ln^n x}{n}$

$q = \lim_{n\to\infty} \frac{\frac{|\ln^{(n+1)} x|}{n+1}}{\frac{|\ln^n x|}{n}} = \lim_{n\to\infty} \frac{|\ln^{n+1} x| \cdot n}{|\ln^n x| \cdot (n+1)} \quad \lim_{n\to\infty} \frac{|\ln^n x \cdot \ln x \cdot n|}{|\ln^n x| (n+1)}$

$$\boxed{|t| > r \Leftrightarrow (t > r \lor t < -r) \qquad r > 0}$$

$$= \lim_{n\to\infty} \frac{|\ln x| \, n}{n+1} = |\ln x| \cdot \lim_{n\to\infty} \frac{n}{n+1} = |\ln x|$$

$$-1 < \ln x < 1$$

$$\ln e^{-1} < \ln x < \ln e \qquad x \in (e^{-1}, e)$$

$$e^{-1} < x < e$$

$$x = e^{-1} \Rightarrow \sum_{n=2}^{\infty} \frac{\ln^n e^{-1}}{n} = \left[\frac{(\ln e^{-1})^n}{n}\right] = \sum_{n=1} \frac{(-1)^n}{n}$$

- Ovaj red konvergira uvjetno.

$$x = e \Rightarrow \sum_{n=2}^{\infty} \frac{\ln^n e}{n} = \sum \frac{1}{n} \quad \text{- divergira}$$

$$S = [e^{-1}, e)$$

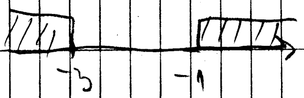
c) $\sum_{n=1}^{\infty} \frac{1}{n\cdot(x+2)^n}$ $\qquad a_n = \frac{1}{n\cdot|x+2|^n}$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{1}{\underbrace{\sqrt[n]{n}}_{1}\cdot|x+2|} = \frac{1}{|x+2|}$$

$$\frac{1}{|x+2|} < 1 \Rightarrow |x+2| > 1$$

$x+2>1$ ili $x+2<-1$ $\qquad x\in(-\infty,-3)\cup(-1,+\infty)$

$x>-1$ ili $x<-3$



Za $x=-3 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n\cdot(-1)^n} = \sum_{n=1}^{\infty} \frac{1}{n(-1)^n}\cdot\frac{(-1)^n}{(-1)^n} = \sum \frac{(-1)^n}{n\cdot\underbrace{(-1)^{2n}}_{1}}$

$= \sum \frac{(-1)^n}{n}$ - red konvergira

$x=-1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n}$ - harmonijski niz divergira

$$S = (-\infty, -3] \cup (-1, +\infty)$$

Za vježbu:

d) $\sum_{n=1}^{\infty} \frac{tg^n x}{n^2+4}$

e) $\sum_{n=1}^{\infty} \frac{n}{\ln^n (x+2)}$

f) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left(\frac{2x}{x^2+1}\right)^n$

(2.) Provjeriti da li dati funkcionalni red konvergira ravnomjerno na datom skupu:

a) $\sum_{n=1}^{\infty} x^{n-1}$, $S = \left[-\frac{1}{2}, \frac{1}{2}\right]$

$S_n(x) = x^0 + x^1 + x^2 + \dots + x^{n-1}$ - suma geometrijskog niza $= \frac{1-x^n}{1-x}$

$= \frac{1}{1-x} - \frac{x^n}{1-x}$

$x \in S \Rightarrow |x| < 1 \Rightarrow x^n \to 0, (n \to \infty)$

$S(x) = \lim_{n \to \infty} S_n(x) = \frac{1}{1-x}$

$$|S_n(x) - S(x)| = \left| \frac{1-x^n}{1-x} - \frac{1}{1-x} \right| = \left| \frac{-x^n}{1-x} \right| = \frac{|x|^n}{1-x} \underset{(*)}{\leq} \left(\frac{1}{2}\right)^n \cdot 2$$

$$= \frac{1}{2^n} \cdot 2 = \frac{1}{2^{n-1}}$$

$$-\frac{1}{2} \leq x \leq \frac{1}{2} \quad | \cdot (-1)$$

$$\frac{1}{2} \geq -x \geq -\frac{1}{2} \quad | + 1$$

$$\frac{1}{2} \leq 1-x \leq \frac{3}{2}$$

$$x \in S \Rightarrow |x| \leq \frac{1}{2}$$

$$1-x \geq \frac{1}{2} \Rightarrow \frac{1}{1-x} \leq 2 \quad (*)$$

$|S_n(x) - S(x)| \leq \frac{1}{2^{n-1}} \to 0$ red konvergira ravnomjerno na S!

b) $$\sum_{n=0}^{\infty} (1-x) \cdot x^n = \sum_{n=0}^{\infty} x^n - x^{n+1} \qquad S = [0,1]$$

$$S_n(x) = \underset{n=0}{(1-x)} + \underset{n=1}{(x-x^2)} + \underset{n=2}{(x^2-x^3)} \ldots \underset{n \leftrightarrow n-1}{(x^{n-1} - x^n)}$$

$$S_n(x) = 1 - x^n \to S_n(x) \begin{cases} 1, & x \neq 1 \text{ tj. } x \in [0,1) \\ 0, & x = 1 \end{cases}$$

$$|S_n(x) - S(x)| = \begin{cases} x^n, & x \in [0,1) \\ 0, & x = 1 \end{cases}$$

$\sup_{x \in S} |S_n(x) - S(x)| = 1$. Ovaj red ne konvergira ravnomjerno na S (sup bi trebao biti 0)

c) $\sum_{n=1}^{\infty} \frac{x}{[(n-1)x+1][nx+1]}$ $\quad S_1 = (\varepsilon, +\infty)$, $\varepsilon > 0$

$S_2 = (0, +\infty)$

$$S_n(x) = \sum_{k=1}^{n} \frac{x}{[(k-1)x+1](kx+1)} =$$

$$(kx+1) - [(k-1)x+1] = kx+1-(k-1)x-1 = kx-kx+x = x$$

$$S_n(x) = \sum_{k=1}^{n} \frac{(kx+1)-[(k-1)x+1]}{[(k-1)x+1][kx+1]} = \sum_{k=1}^{n} \left[\frac{kx+1}{[(k-1)x+1](kx+1)} - \frac{(k-1)x+1}{[(k-1)x+1][kx+1]} \right.$$

$$= \sum_{k=1}^{n} \left[\frac{1}{(k-1)x+1} - \frac{1}{kx+1} \right] = 1 - \frac{1}{x+1} + \frac{1}{x+1} - \frac{1}{2x+1} + \frac{1}{2x+1} + \dots + \frac{1}{(n-1)x+1}$$

$$- \frac{1}{nx+1} = 1 - \frac{1}{nx+1}$$

$S(x) = 1$ za $x \in S_1$ i $x \in S_2$

$$|S_n(x) - S(x)| = \left|\cancel{1} - \frac{1}{nx+1} - \cancel{1}\right| = \frac{1}{nx+1}$$

U skupa S_1: $x \in S_1 \Rightarrow x > \varepsilon \Rightarrow nx+1 > n\varepsilon+1 \Rightarrow$

$$\frac{1}{nx+1} < \frac{1}{n\varepsilon+1}$$

$$x \in S_1 \Rightarrow |S_n(x) - S(x)| < \frac{1}{n\varepsilon+1} \to 0 \quad (n \to \infty)$$

$$S_n(x) \underset{S_1}{\rightrightarrows} S(x)$$

$\sup_{x \in S_2} |S_n(x) - S(x)| = \underline{\underline{1}}$ $\to$ red ne konvergira ravnomjerno na skupu S_2.

Za vježbu

d) $\sum_{n=1}^{\infty}\left(\frac{x^{n-1}}{n}-\frac{x^{n}}{n+1}\right)$ e) $\sum_{n=1}^{\infty}\frac{1}{(x+n)(x+n+1)}$

$I=[-1,1]$ $I=[0,+\infty)$

(3.) Dokazati pomoću Vajerštrasovog kriterija konvergenciju datog reda na datom skupu!

a) $\sum_{n=1}^{\infty}\frac{\cos nx}{n^{\alpha}}$, $\alpha>1$, $x\in\mathbb{R}$

$\left|\frac{\cos nx}{n^{\alpha}}\right|<\frac{1}{n^{\alpha}}$, red $\sum\frac{1}{n^{\alpha}}$ - konvergira za $\alpha>1$

Prema Vajerštrasovom kriteriju imamo konvergenciju

b) $\sum_{n=1}^{\infty}\frac{1}{x^{2}+n^{3}}$, $x\in\mathbb{R}$

$\frac{1}{x^{2}+n^{3}}<\frac{1}{n^{3}}$

$\sum_{n=1}^{\infty}\frac{1}{n^{3}}$ - konvergira

c) $\sum_{n=1}^{\infty}\underbrace{\frac{\operatorname{arctg}(n^{n}x+2x^{3})\cdot\cos n\pi x}{n\sqrt{n}}}_{fn(x)}$

$|fn(x)|<\frac{\frac{\pi}{2}}{n\sqrt{n}}=\frac{\frac{\pi}{2}}{n^{\frac{3}{2}}}$ $\sum\frac{\frac{\pi}{2}}{n^{\frac{3}{2}}}$ - konvergira

$$\boxed{|\cos x|<0}$$

d) $\sum_{n=1}^{\infty} \frac{nx}{1+n^5x^2}$, $x \in \mathbb{R}$

$f_n(x) = \frac{nx}{1+n^5x^2}$

D.P. $x \in \mathbb{R}$

- funkcija je neparna
- Nula : $x=0$
- Znak $x>0 \Rightarrow f_n(x)>0$

  $x<0 \Rightarrow f_n(x)<0$

$\lim_{x\to\pm\infty} f_n(x) = 0$ H.A. $y=0$

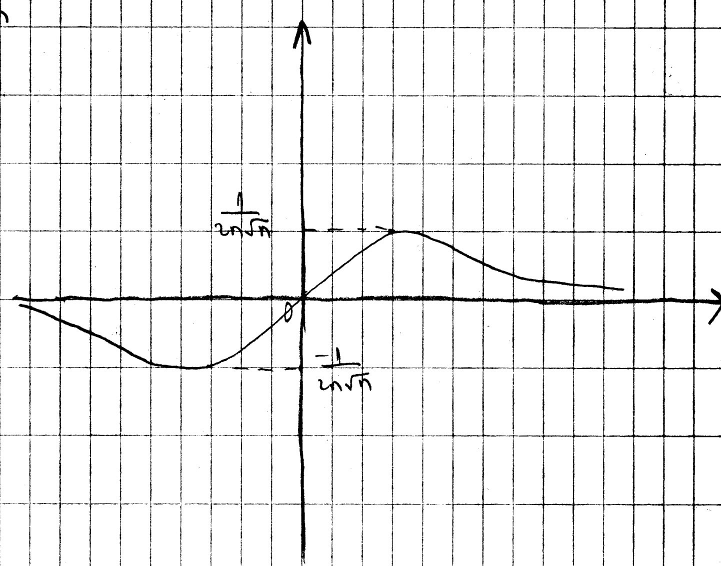
$f_n'(x) = \frac{n(1+n^5x^2) - nx\cdot 2n^5\cdot x}{(1+n^5x^2)^2} = \frac{n+n^6x^2-2n^6x^2}{(1+n^5x^2)^2} = \frac{n-n^6x^2}{(1+n^5x^2)^2}$

$f_n'(x)=0 \Rightarrow n=n^6x^2 \Rightarrow x^2=\frac{1}{n^5} \Rightarrow x=\pm\frac{1}{\sqrt{n^4\cdot n}} = \pm\frac{1}{n^2\sqrt{n}}$

	$-\infty$	$-\frac{1}{n^2\sqrt{n}}$		$\frac{1}{n^2\sqrt{n}}$		$+\infty$
$f_n'(x)$	$-$		$+$		$-$	
$f_n(x)$	$\searrow$		$\nearrow$		$\searrow$	
		min		max		

$f_n\left(\frac{1}{n^2\sqrt{n}}\right) = \frac{n\cdot\frac{1}{n^2\sqrt{n}}}{1+n^5\cdot\frac{1}{n^5}} = \frac{1}{2n\sqrt{n}}$

$f_n\left(-\frac{1}{n^2\sqrt{n}}\right) = -\frac{1}{2n\sqrt{n}}$



$$|f_n(x)| \leq \frac{1}{2n\sqrt{n}} \quad (x \in \mathbb{R})$$

$$\sum \frac{1}{2n\sqrt{n}} = \sum \frac{1}{2n^{\frac{3}{2}}}$$ – dati red konvergira ravnomjerno!

e) $$\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n!}}(x^n + x^{-n}) \qquad \frac{1}{2} \leq x \leq 2$$

$$x^n = t \implies x^{-n} = \frac{1}{t}$$

$$x^n + x^{-n} = t + \frac{1}{t} = \frac{t^2+1}{t}, \quad \frac{1}{2^n} \leq t \leq 2^n \quad (n \in \mathbb{N})$$

$$g(t) = \frac{t^2+1}{t}$$

$$g'(t) = \frac{2t^2 - (t^2+1)}{t^2} = \frac{t^2-1}{t^2} = \frac{(t-1)(t+1)}{t^2}$$

$$t > 1 \implies g'(t) > 0 \implies g \nearrow$$

$$\sup_{x \in [\frac{1}{2}, 2]} (x^n + x^{-n}) = 2^n + 2^{-n}$$

$$f_n(x) \leq \frac{n^2}{\sqrt{n!}}(2^n + 2^{-n}) < \frac{n^2}{\sqrt{n!}}(2^n + 2^n) = \frac{n^2}{\sqrt{n!}} \cdot 2 \cdot 2^n = \frac{n^2}{\sqrt{n!}} \cdot 2^{n+1}$$

Posmatramo red $\sum_{n=1}^{\infty} a_n \quad a_n = \frac{n^2}{\sqrt{n!}} \cdot 2^{n+1}$

$$q = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^2}{\sqrt{(n+1)!}} \cdot 2^{n+2}}{\frac{n^2}{\sqrt{n!}} \cdot 2^{n+1}} = \lim_{n\to\infty} \frac{(n+1)^2 \cdot 2^{n+1} \cdot 2 \cdot \sqrt{n!}}{n^2 \cdot 2^{n+1} \cdot \sqrt{n!(n+1)}}$$

$$= \underbrace{\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2}_{1} \cdot 2 \cdot \underbrace{\lim_{n\to\infty} \frac{1}{\sqrt{n+1}}}_{0} = 0$$

$q < 1 \implies$ red $\sum a_n$ po Vajerštrasovom kriteriju ravnomjerno konvergira

Za vježbu:

f) $\sum_{n=1} \frac{x}{1+n^4x^3}$, $x \in [0, +\infty]$

radit kao pod d

g) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{(3n+1)\cdot 3^n}$, $-1 \le x \le 3$

h) $\sum_{n=1}^{\infty} \frac{n \ln(1+nx)}{x^n}$, $x \in [1+\alpha, +\infty)$ $\alpha > 0$

i) $\sum_{n=1}^{\infty} \sin\frac{1}{nx} \cdot \ln\left(1+\frac{x}{\sqrt{n}}\right)$, $x \in (0, +\infty)$

zadaci s rokova:

$$A = \int \frac{x+1}{x(1+xe^x)}\,dx \;\begin{matrix}\cdot e^x \\ \cdot e^x\end{matrix} = \int \frac{e^x(x+1)}{e^x \cdot x(1+xe^x)}\,dx = \int \frac{xe^x + e^x}{xe^x(1+xe^x)}\,dx$$

$$\left| \begin{matrix} 1+xe^x = t \\ (e^x + xe^x)\,dx = dt \end{matrix} \right| = \int \frac{dt}{t \cdot (t-1)}$$

$$\frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{t-1} = \ldots$$

$$I = \int e^{\arcsin x}\,dx = \left| \begin{matrix} u = e^{\arcsin x} & dv = dx \\ du = e^{\arcsin x} \cdot (\arcsin x)' & v = x \\ du = e^{\arcsin x} \cdot \frac{1}{\sqrt{1-x^2}} & \end{matrix} \right|$$

$$I = x \cdot e^{\arcsin x} - \underbrace{\int \frac{x \cdot e^{\arcsin x}}{\sqrt{1-x^2}}\,dx}_{I_1}$$

$$J_1 = \int \frac{x \cdot e^{\arcsin x}}{\sqrt{1-x^2}}\,dx \quad \left| \begin{array}{l} x = \sin t \\ \arcsin x = t \\ \frac{1}{\sqrt{1-x^2}}\,dx = dt \end{array} \right|$$

$$= \int \sin t \cdot e^t\,dt = \left| \begin{array}{ll} \sin t = u & dv = e^t \\ \cos t\,dt = du & v = e^t \end{array} \right|$$

$$= e^t \cdot \sin t - \int e^t \cos t\,dt = \left| \begin{array}{ll} u = \cos t & dv = e^t \\ du = -\sin t\,dt & v = e^t \end{array} \right|$$

$$J_1 = e^t \sin t - e^t \cos t - J_1$$

$$2 J_1 = e^t (\sin t - \cos t)$$

$$J_1 = \frac{e^t}{2} (\sin t - \cos t)$$

$$J = x \cdot e^{\arcsin x} - \frac{e^t}{2} (\sin t - \cos t)$$

FUNKCIONALNI REDOVI (ponovo!)

1. Naći oblast konvergencije reda

$$\sum_{n=1}^{\infty} e^{-nx}$$

$$\lim_{n\to\infty} \sqrt[n]{|e^{-nx}|} = \lim_{n\to\infty} \sqrt[n]{|e^{-x}|^n} = e^{-x}$$

$e^{-x} < 1 = e^0$

$-x < 0 \Rightarrow \boxed{x > 0} \Rightarrow x \in (0, +\infty)$ – oblast konvergencije

$x = 0 \Rightarrow \sum e^0 = \sum_{n=1}^{\infty} 1$ – divergira (opći član $\not\to 0$)

b) $\sum_{n=1}^{\infty} \frac{\ln^n x}{n}$

$$q = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$$

$$a_n = \left| \frac{\ln^n x}{n} \right|$$

$$q = \lim_{n\to\infty} \frac{\left|\frac{\ln^{n+1} x}{n+1}\right|}{\left|\frac{\ln^n x}{n}\right|} = \lim_{n\to\infty} \frac{|\ln^{n+1} x| \cdot n}{(n+1)|\ln^n x|} = \lim_{n\to\infty} \frac{|\cancel{\ln^n x}|\,|\ln x|\, n}{(n+1)|\cancel{\ln^n x}|}$$

$$= \lim_{n\to\infty} \frac{|\ln x| \cdot n}{n+1} = |\ln x|$$

II način: $\lim_{n\to\infty} \sqrt[n]{\frac{\boxed{|\ln^n x|}}{n}} = \frac{|\ln x|}{\underbrace{\sqrt[n]{n}}_{1}} = |\ln x|$

$|\ln x| < 1$

$-1 < \ln x < 1$

$\ln e^{-1} < \ln x < \ln e$

$\Rightarrow e^{-1} < x < e \qquad x \in \left(\frac{1}{e}, e\right)$

$x = e \Rightarrow \ln x = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n}$ - harmonijski red divergira

$x = \frac{1}{e} \quad \ln x = -1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ ← konvergira

II Provjerite da li dati funkcionalni red konvergira ravnomjerno na datom skupu po definiciji:

a) $\sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + \dots \qquad T = \left[-\frac{1}{2}, \frac{1}{2}\right]$

$S_n(x) = 1 + x + x^2 + \dots + x^{n-1} = \frac{1-x^n}{1-x}$

$\lim_{n\to\infty} S_n(x) = \frac{1}{1-x} \qquad$ (jer $x^n \to 0$)

$|S_n(x) - S(x)| = \left|\frac{1-x^n}{1-x} - \frac{1}{1-x}\right| = \left|\frac{1-x^n-1}{1-x}\right| = \left|\frac{-x^n}{1-x}\right| =$

$= \frac{|x|^n}{1-x} \qquad \frac{|x|^n}{1-x} \leq \frac{\left(\frac{1}{2}\right)^n}{\frac{1}{2}} = 2 \cdot \left(\frac{1}{2}\right)^n \to 0$

radi $\left[-\frac{1}{2}, \frac{1}{2}\right]$
ne treba | |

$-\frac{1}{2} \leq x \leq \frac{1}{2} \quad | \cdot (-1) \qquad \frac{1}{2} \leq x+1 \leq \frac{3}{2} \quad |^{-1}$

$-\frac{1}{2} \leq -x \leq \frac{1}{2} \quad | +1 \qquad \frac{3}{2} \leq \frac{1}{1-x} \leq$

b) $\sum_{n=0}^{\infty} \underbrace{(1-x)x^n}_{f_n(x)}$ $T=[0,1]$

$f_n(x)=x^n-x^{n+1}$

$S_n(x)=1-x+x-x^2+x^2-x^3+\cdots+x^{n-1}-x^n \quad =1-x^n$

$S(x)=\lim_{n\to\infty} S_n(x)=\begin{cases} 1, & x\in[0,1) \\ 0 & x=1 \end{cases}$

$|S_n(x)-S(x)|=\begin{cases} x^n & , \quad 0\le x<1 \\ 0 & , \quad x=1 \end{cases}$

$\sup_{x\in[0,1)} |S_n(x)-S(x)|=1 \quad \Leftarrow$ red ne konvergira ravnomjerno!

③ Dokazati da konvergira ravnomjerno dati red na datom skupu pomoću Vajerstrasovog kriterija:

a) $\sum_{n=1}^{\infty} \frac{\cos nx}{n^\alpha}$, $\alpha>1$, $T=R$

$\left|\frac{\cos nx}{n^\alpha}\right| \le \frac{1}{n^\alpha}$; $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ konvergira $\Rightarrow$ dati red ravnomjerno konvergira prema Vajerštrasu.

b) $\sum_{n=1}^{\infty} \frac{1}{x^2+n^3}$ $\qquad T = \mathbb{R}$

$\left|\frac{1}{n^2+n^3}\right| \le \frac{1}{n^3}$, $\sum_{n=1}^{\infty} \frac{1}{n^3}$ konvergira !

c) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{(3n+1)3^n}$, $-1 \le x \le 3$ $\qquad$ ($T = [-1, 3]$

$-1 \le x \le 3 \quad | -1$

$-2 \le x-1 \le 2 \quad \Rightarrow \quad |x-1| \le 2$

$$\left|\frac{(x-1)^n}{(3n+1)\,3^n}\right| \le \frac{2^n}{(3n+1)\,3^n}$$

Prema D'Alambert-ovom kriteriju

$$\lim_{n\to\infty} \frac{\frac{2^{n+1}}{(3(n+1)+1)\cdot 3^{n+1}}}{\frac{2^n}{(3n+1)\,3^n}} = \lim_{n\to\infty} \frac{2^{n+1}\cdot \cancel{3^n}\cdot (3n+1)}{\cancel{2^n}\cdot(3n+4)\cdot 3^{n+1}} =$$

$$= \lim_{n\to\infty} \frac{2\,(3n+1)}{(3n+4)\cdot 3} = \frac{2}{3}$$

$\Rightarrow$ red $\sum \frac{2^n}{(3n+1)\cdot 3^n}$ konvergira

d) $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{nx}{1+n^5x^2}}_{f_n(x)}$, $T=\mathbb{R}$

$$f_n'(x) = \left(\frac{nx}{1+n^5x^2}\right)' = \frac{n\cdot(1+n^5x^2) - nx\cdot 2n^5x}{(1+n^5x^2)^2}$$

$$= \frac{n+n^6x^2-2n^6x^2}{(1+n^5x^2)^2} = \frac{n-n^6x^2}{(1+n^5x^2)^2}$$

$f_n'(x)=0$

$n-n^6x^2=0$

$n=n^6x^2$

$$x^2=\frac{n}{n^6}=\frac{1}{n^5}$$

$$x_{1,2}=\pm\sqrt{\frac{1}{n^5}}=\pm\frac{1}{n^2\sqrt{n}}$$

x	$-\infty$		$-\frac{1}{n^2\sqrt{n}}$		$\frac{1}{n^2\sqrt{n}}$		$+\infty$
$f_n'(x)$		$-$	0	$+$	0	$-$	
$f_n(x)$		$\searrow$	min	$\nearrow$	max	$\searrow$	

$$T_{max}\left(\frac{1}{n^2\sqrt{n}}, \frac{1}{2n\sqrt{n}}\right)$$

$$T_{min}\left(-\frac{1}{n^2\sqrt{n}}, -\frac{1}{2n\sqrt{n}}\right)$$

4) $\displaystyle\sum_{n=1}^{\infty} \sin\frac{1}{nx}\ln\left(1+\frac{x}{n}\right)$, $T=(0,+\infty)$

$\sin x < x$ pt $x\in\left(0,\frac{\pi}{2}\right)$

$\sin\frac{1}{nx} < \frac{1}{nx}$ pt $x>0$, $n\in\mathbb{N}$

$$\frac{1}{n+1} < \ln\left(1+\frac{1}{n}\right) < \frac{1}{n} \quad (n\in\mathbb{N})$$

$\ln(1+t) < t \qquad (t>0)$

$\ln\left(1+\frac{x}{n}\right) < \frac{x}{n} \quad , \qquad x>0$

$\sin\frac{1}{nx} \cdot \ln\left(1+\frac{x}{n}\right) < \frac{1}{nx}\cdot\frac{x}{n} = \frac{1}{n^2}$ red $\sum_{n=1}^{\infty} \frac{1}{n^2}$ konvergira

Za vježbu:

a) $\sum_{n=1}^{\infty} 2^{-n} \cdot \cos n\pi x \quad , \qquad T=\mathbb{R}$

b) $\sum_{n=1}^{\infty} \frac{\operatorname{arctg}(n^2 x) \cos n\pi x}{n\sqrt{n}} \quad , \qquad T=\mathbb{R}$

c) $\sum_{n=1}^{\infty} \underbrace{\frac{n^2}{n+1} \cdot \frac{x^2 \cdot \sin x}{1+n^5 x^4}}_{f_n(x)}$

uputa:

$g(x) = \frac{x^2}{1+n^5x^4}$ parna, DP $\forall x\in\mathbb{R}$) H.A. $x=1$

Dokazati da je $|g(x)| \le \frac{1}{2n^2\sqrt{n}} \quad (x\in\mathbb{R})$

$g'(x) = \dots$

$\Rightarrow |f_n(x)| \le \dots$

d)